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Measuring Distance-to-Default for Financial and Non-Financial Firms

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Editorial Comment: Data for DTD* is available for all listed companies under the coverage of the Credit Research Initiative at the Risk Management Institute of NUS. They can be downloaded from www.rmicri@org.

INTRODUCTION

This article reviews several empirical methodologies for estimating Distance-to-Default (DTD), a popular measure for gauging how far a limited-liability firm is away from default. We focus on the idea behind each method and discuss its strengths and weaknesses both conceptually and through the use of concrete examples. The methodological differences and implications are brought to the fore by analyzing several banks and insurance companies, which are typically of high financial leverage. We show that distortion can be substantial when an inappropriate estimation method is applied.

DTD has been widely adopted by academics in financial research and extensively used in business applications by industry practitioners. The academic papers that use DTD are too numerous to list here whereas in the commercial applications, Moody's KMV model is arguably the most prominent one. The precise definition of DTD depends on a theoretical model, particularly the seminal credit risk model of Merton (1974), which treats corporate debt as an option-like financial instrument. Conceptually, a firm's asset value evolves according to some stochastic dynamic and its debt will be honored when the asset value stays above the promised payment in some future time stipulated under the debt contract. Otherwise, this firm is in default and its debt holders can only recover a partial amount equal to what is left of the firm.

Even though the future asset value of a firm cannot be known today, its current value serves as the natural base with which one can assess how likely the firm will default in the future. For example, when the current firm value is much higher than its promised future payments, the likelihood of default will be small simply because the firm has significant buffer to absorb losses in its asset value. This thinking underlies how we traditionally view corporate financial leverage, such as the debt-to-asset ratio. A lower-leverage firm is expected to be more resilient to future losses. Since the asset value moves randomly due to external shocks, the leverage ratio alone cannot be good enough to adequately capture the notion of DTD. The trend and volatility of the asset value movements must play an important role in determining the likelihood of default, because the same level of buffer may not be sufficient to withstand potential losses when the firm's asset value is highly volatile. Put simply, a good DTD must be a leverage ratio adjusted for trend and volatility of the firm's asset value.

We introduce the Merton (1974) model to define such a DTD. Although DTD is an appealing concept, it runs into two kinds of implementation challenges. Computing DTD requires knowing the market value of the firm's assets and the parameters governing the asset value movements (trend and volatility). But the market value of a firm's assets as postulated in the Merton (1974) model cannot be directly observed. Without a time series of observed asset values, it is obviously difficult to estimate the model parameters that define trend and volatility of the asset value movements. Different estimation methodologies have been proposed in the literature; for example, (1) the market value proxy method used in Brockman and Turtle (2003) and Eom, Helwege and Huang (2004), among others, (2) the volatility restriction method proposed by Jones, Mason and Rosenfeld (1984) and Ronn and Verma (1986), (3) the KMV iterative method described in Crosbie and Bohn (2003), and (4) the transformed-data maximum likelihood method by Duan (1994, 2000). We describe these methodologies and discuss their strengths and weaknesses with concrete examples. Specifically, we use financial firms to illustrate the limitation of the KMV estimation method. Because financial firms typically have a large proportion of liabilities that cannot be accounted for by the KMV estimation method (for example, policy obligations of an insurance company), the method tends to inflate asset volatility and cause a distortion to DTD. We argue that the maximum likelihood method proposed first by Duan (1994) and modified later by Duan (2010) and Duan *et al*. (2012) to deal with financial firms is the most appropriate and flexible method for estimating DTD.

The second application challenge arises from applying DTD strictly according to the structural credit risk model as defined. The Merton (1974) model and many subsequent models along the same line are typically classified in the literature as structural credit risk models in contrast to reduced-form models. Strictly following the Merton (1974) model, one can obtain a firm's default probability by directly applying the cumulative normal distribution to the negative of DTD. But the results from such a direct and consistent application of the structural model are at odds with empirical default rates. Academic researchers and industry practitioners have long realized that DTD is highly informative about defaults, but it must be used along with other variables to achieve good performance.[1] Further calibration through a reduced-form model, such as logistic regression, is a must in practice. It is somewhat ironic to say that DTD, as a measure defined by a structural credit risk model, must be further calibrated by a reduced-form model to yield good empirical performance. How DTD can be intelligently applied is not within the scope of this article. We bring up this issue so that readers can have a general awareness of the limitation of applying DTD strictly in accordance with the Merton (1974) model even though DTD is built upon that model.

I. THE DISTANCE-TO-DEFAULT

The Merton (1974) model assumes that firms are financed by equity, with its value at time t denoted by S_t, and one single pure discount bond (denoted by D_t) with maturity date T and principal F. The asset value V_t follows a geometric Brownian motion:

$$dV_t = \mu V_t dt + \sigma V_t dW_t. \tag{1}$$

Here, W_t is a standard Brownian motion. Due to limited liability, the equity value at maturity is $S_T = \max(V_T - F, 0)$. Therefore, the equity value at time $t \leq T$ by the Black-Scholes option pricing formula becomes

$$S(V_t, \sigma) = V_t N(d_t) - e^{-r(T-t)} F N\left(d_t - \sigma\sqrt{T-t}\right), \tag{2}$$

where r is the instantaneous risk-free rate, $N(.)$ is the standard normal cumulative distribution function, and

$$d_t = \frac{\ln\left(\frac{V_t}{F}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{3}$$

Following the Merton (1974) model, it can be shown that the probability of the company's default at time T evaluated at time t is $N(-\text{DTD}_t)$ where the DTD at time t is defined as

$$\text{DTD}_t = \frac{\ln\left(\frac{V_t}{F}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}. \quad (4)$$

Since the standard normal distribution function is universal, the sole factor that determines the default probability is the DTD. As the formula suggests, DTD is the logarithm of the leverage ratio shifted by the expected return $(\mu - \sigma^2/2)(T - t)$, and scaled by the volatility $\sigma\sqrt{T-t}$.

Consider two firms with identical leverage ratios and volatilities, but the asset value of one is expected to increase at a faster rate than the other. We naturally expect the one with a higher expected return to be further away from default, i.e., have a larger DTD. If two firms have identical leverage ratios and expected returns, their volatilities will determine which one is farther away from default. It is evident that the conclusion depends on the sign of the numerator. If the numerator is positive, meaning that the asset value will cover the debt obligation on average, a lower volatility should make the firm less likely to default, and it indeed has a larger DTD. When the numerator is negative, the situation can be understood as the firm is on average not expected to meet its debt obligation in the future. A higher volatility will make DTD less negative, which is consistent with the intuition that the firm has a higher chance, due to a higher volatility, to get its future asset value to exceed the debt obligation.

The default probability of interest is naturally the one under the probability law that governs the physical movement of the asset value; that is what the DTD formula in Equation (4) intends to capture. In the credit risk literature, the risk neutral probability often surfaces to describe a hypothetical scenario in which economic agents are risk neutral. Were economic agents indeed risk neutral, the expected return on financial investment would have to equal the risk-free rate of return. By implication, parameter μ in Equation (4) should be replaced by r under the hypothetical scenario. Computing the risk-neutral DTD and then the risk-neutral default probability does not need the value of μ. Moreover, the volatility parameter, σ can be obtained by just calibrating a pricing model to the observed market price of debt and/or prices of some credit derivatives. Obviously, the risk-neutral default probability is easier to obtain. However, one should be mindful of the fact that in theory it is not the default probability to be physically experienced. Since it is the physical DTD that we are after in practice, we should have suitable estimates for both expected return and volatility.

Due to the nature of diffusion models, however, parameter μ cannot be estimated with reasonable precision using high frequency data over a time span of several years, a well-known fact in the financial econometrics literature. The technical reason for this result is that parameter μ in Equation (1) is accompanied by a time factor of dt whereas parameter σ is by a time factor of $\sqrt{dt}$ which is implicit in dW_t. Data sampled frequently is less informative about μ than σ, because dt is much smaller than $\sqrt{dt}$ when the value of dt is small. With the estimation precision issue in mind, it makes empirical sense to avoid using μ in the DTD estimation, particularly when DTD is only used as an input to a reduced-form model to be further calibrated to empirical default rates. Therefore, it may be advisable to use the following alternative form of DTD to reduce sampling errors:

$$\text{DTD}_t^* = \frac{\ln\left(\frac{V_t}{F}\right)}{\sigma\sqrt{T-t}} \quad (5)$$

Note that DTD* amounts to setting $\mu = \sigma^2/2$ in Equation (4), and its calculation does not require the value of μ.[2] As shown later in this paper, estimated DTD* is much more stable than DTD.

II. ESTIMATION METHODS

There are several difficulties in implementing the Merton (1974) model. First, the asset values are not directly observable, and therefore they are not available for plugging into the DTD formula directly even if the model parameters were readily available. Second, the parameters μ and σ governing the unobserved asset value process are unknown and need to be estimated. But their estimation becomes a serious challenge simply because the asset values are not directly observed. Several estimation methods are commonly applied in the finance literature and in business practice, but not enough attention is paid to their theoretical and empirical shortcomings. We introduce these methods through the use of concrete examples and discuss their limitations and shortcomings. Our discussions specifically touch upon a new methodological advancement for dealing with financial firms which factors in their somewhat unique liability structure.

We use three different types of firms in our empirical illustration. They are IBM (a US industrial firm), Barclays (a British bank) and Tokio Marine (a Japanese insurance company). These three firms are used for general comparison of different estimation methods. Later, we will focus on financial firms to zero in on the difference between the KMV method and the transformed-data maximum likelihood method. For that, we use three banks (Bank of America, Barclays, and DBS) and three insurance companies (Sun Life, AXA SA and Tokio Marine) from different regions of the world so as to appreciate that the methodological impact is far reaching.

2.1. The Market Value Proxy Method

It is easy to obtain the equity value of an exchange listed firm, but the same cannot be said about the asset value. A direct valuation of asset value is practically impossible, because a firm as a going concern presumably possesses intangible assets and their values are hard to determine. Adding together the market values of equity and debt to arrive at the market value of the firm makes sense conceptually, but the market value of debt is hard to come by because a typical firm will have a large portion of debt in some non-tradable forms. Thus, a hybrid approach of adding market capitalization (equity value) to the book value of liabilities became very popular in corporate finance literature. The papers using this market value proxy method to obtain firm value are too numerous to mention. If one adopts the market value proxy method, estimating the two parameters (μ and σ) becomes fairly straightforward. One can obtain a time series of, say, daily asset values by summing daily updated market capitalizations with quarterly updated book values of liabilities. With the time series in place, one can obtain the daily logarithmic asset returns and then compute the sample mean and standard deviation of the return as the estimates for μ and σ. Indeed, the market value proxy method has been used in the empirical credit risk literature as well, for example, Brockman and Turtle (2003) and Eom *et al*. (2004).

However, the quality of the market value proxy method is questionable. It has been argued in Wong and Choi (2009) that such a method will produce an upward biased estimate of the asset value. In fact, the bias magnitude is directly related to how volatile the firm's asset value is. The reason is not difficult to appreciate. When the value of a discount debt is artificially set to its par value, it has inflated the market value and the amount by which it has been inflated (the discount portion) increases with the firm's volatility, by standard option pricing theory. Its impact on credit analysis has not, in our opinion, been fully appreciated in the literature. For example, the market value proxy makes the firm's equity (as a call option) always in the money at the time of assessing credit risk. If other things are equal, the DTD will be biased upwards, making the estimated default probability smaller than it should be. In light of this, it is unclear as to how one should interpret the empirical findings in, say, Eom *et al*. (2004).

To illustrate the market value proxy method, we consider three firms in different sectors and countries — IBM, Barclays, and Tokio Marine. The input variables used to estimate DTD and DTD* are given in Panel A of Table 1. Their values are as of the end of December 2011. Note that the last entry is equity volatility. It is not needed for the market value proxy method, but will be used for the method discussed in the next section. For IBM, Barclays and Tokio Marine, the values (except for equity volatility) are in million USD, million GBP and million JPY, respectively. In order to compute μ and σ, we use one year of market capitalizations on a daily basis and add quarterly updated book value of liabilities to form one year's worth of daily asset value time series. We then compute the sample mean and standard deviation of the continuously compounded daily returns. The values for μ and σ are annualized in the typical fashion. With the

Table 1. Different estimation methods on three types of firms.

	IBM (Million USD)	Barclays (Million GBP)	Tokio Marine (Million JPY)
Panel A: Input Variables			
Market cap	216,724	21,477	1,371,714
Short-term debt	39,843	255,193	1,922,395
Long-term debt	21,915	171,657	121,673
Other liabilities	28,506	1,004,083	12,095,019
Equity volatility	22.44%	56.15%	31.56%
Panel B: The Market Value Proxy Method			
μ	14.91%	−6.94%	−7.53%
σ	16.06%	6.67%	4.84%
Asset value (12/2011)	306,988	1,452,410	15,510,801
DTD (12/2011)	8.4697	−0.8495	0.3326
DTD* (12/2011)	7.6215	0.2233	1.9147
Panel C: The Volatility Restriction Method			
μ	17.03%	−7.69%	−27.51%
σ	18.19%	3.45%	12.74%
Asset value (12/2011)	278,482	448,327	3,415,782
DTD (12/2111)	10.2009	5.6874	2.0445
DTD* (12/2011)	9.2645	7.8173	4.2032
Panel D: The KMV Method			
μ	17.09%	−7.90%	−26.90%
σ	18.51%	6.09%	16.84%
Asset value (12/2011)	267,464	359,291	3,352,857
DTD (12/2011)	9.8056	−0.4710	1.4360
DTD* (12/2011)	8.9748	0.8563	3.1174
Panel E: The Transformed-Data MLE Method (with the KMV assumption)			
μ	13.06%	−1.58%	−20.33%
σ	18.47%	6.91%	16.83%
Asset value (12/2011)	267,464	358,293	3,352,858
DTD (12/2011)	9.6054	0.4517	1.8285
DTD* (12/2011)	8.9911	0.7148	3.1208
Panel F: The Transformed-Data MLE Method (including other liabilities)			
μ	10.70%	−1.02%	−5.08%
σ	18.02%	1.54%	5.17%
δ	45.78%	61.83%	60.78%
Asset value (12/2011)	280,498	979,658	10,696,331
DTD (12/2011)	8.7478	0.5292	1.6400
DTD* (12/2011)	8.2132	1.1915	2.6339

parameter values in place, we then compute the DTD and DTD* for the end of December 2011 according to Equations (4) and (5). The calculation assumes the maturity equal to one year and the debt amount equal to the total liabilities at the December end. The results are reported in Panel B of Table 1.

2.2. The Volatility Restriction Method

A popular way of implementing the Merton (1974) model for pricing corporate bonds and other credit sensitive instruments is the volatility restriction method of Jones *et al*. (1984) and Ronn and Verma (1986). Here, we present the version by Ronn and Verma (1986) which has been widely applied in deposit insurance literature along with the Merton (1977) deposit insurance pricing model. The method has also been applied in the empirical credit risk literature such as Hillegeist *et al*. (2004) and Campbell *et al*. (2008) without recognizing that their estimation method is in essence that of Ronn and Verma (1986).

The volatility restriction method uses the following two-equation system:

$$S_t = S(V_t;\sigma) \quad (6)$$

$$\sigma_{S_t} = \sigma \frac{V_t}{S(V_t;\sigma)} N(d_t) \quad (7)$$

where $S(V_t;\sigma)$, $N(.)$ and d_t have already been given in Equations (2) and (3), and σ_{S_t} is equity volatility. Equation (6) links the observed market capitalization to its theoretical counterpart implied by the model. Equation (7) forms a volatility restriction linking the equity volatility to the asset volatility where its right-hand side can be derived by applying Ito's lemma to the pricing formula in Equation (2). There are two unknowns in the above two equation system — V_t and σ. With the two equations, one can proceed to solve for the two unknowns.[3]

In pricing applications, the volatility restriction method is a calibration which can be carried out just for one time point. Once V_t and σ are available, one can compute the prices of contingent claims. For the DTD, however, one must also know μ. There are two ways of obtaining such an estimate. First, one can repeatedly solve the two-equation system to obtain asset values for many time points, and then compute the sample mean of continuously compounded returns derived from these implied asset values to obtain an estimate for μ. Second, one can solve the two-equation system only once at the time of interest, and apply the obtained asset volatility to all earlier time points to obtain the implied asset values. These implied asset values will differ from those obtained by the first method, but can similarly be used to produce an estimate for μ. Here, we apply the second approach to obtain μ.

We apply the volatility restriction estimation method to produce estimates for the three firms as of the end of December 2011. We assume that the debt maturity is one year and the default point (i.e., effective debt level for triggering default) is as in the KMV method (to be discussed in the next section). Since the KMV default point is always lower than the total liabilities, this assumption alone will cause the DTD and other estimates to differ from those produced by the market value proxy method. This choice of default point is made to facilitate comparisons with the KMV and other methods to be introduced later.

Equity volatility appears in the left-hand side of Equation (7), and its value is per usual estimated by computing the sample standard deviation of the continuously compounded equity returns over some sample period. For this, we use a one-year long time series of daily equity returns to estimate equity volatility, an input to the volatility restriction method. The equity volatilities for the three firms are given in Panel A of Table 1.

For estimating parameter μ, we use the estimated asset volatility corresponding to the last data point of the one-year long time series of daily market capitalizations to solve for the entire time series of implied asset values. Our results are reported in Panel C of Table 1. It is clear that the results are quite different from those produced under the market value proxy method. This is of course not at all surprising due to at least two factors: (1) the volatility restriction method recognizes the optionality of corporate liabilities but the market value proxy method does not, and (2) the volatility restriction method has been implemented with the KMV default point assumption that only factors in parts of the total liabilities.

Duan (1994) pointed out a critical methodological problem with the volatility restriction method. In essence, the volatility restriction in Equation (7) is obtained by applying stochastic differentiation to the pricing formula in Equation (2). According to the Merton (1974) model, the derived equity volatility must be a stochastic variable. Since it is not a parameter, the sample standard deviation of equity returns should not be the quantity being plugged into the left-hand side of Equation (7). Conceptually, it cannot provide an additional restriction for identification. In practice, one can still obtain estimates as shown in this section, but abusing the system could produce seriously biased estimates as was demonstrated by, for example, Ericsson and Reneby (2005) using a simulation study.

2.3. The KMV Method

The KMV method powers the credit analytics service offered by Moody's KMV. The method is described in reasonable detail in Crosbie and Bohn (2003). The method is sometimes mistakenly understood to be the volatility restriction method described above.[4] Crosbie and Bohn (2003) started out describing the KMV method on pages 13 and 16 as if it were the volatility restriction method but without a reference to Ronn and Verma (1986). Later on page 17, they described the actual KMV method as an iterative procedure consisting of the following steps:

Step 1: Apply an initial value of σ to Equation (5) to obtain a time series of implied asset values and hence continuously compounded asset returns.
Step 2: Use the time series of continuously compounded asset returns to obtain updated estimates for μ and σ.
Step 3: Go back to Step 1 with the updated σ unless convergence has been achieved.

The KMV implementation fixes the maturity at one year and sets the default point to the sum of the short-term debt and one half of the long-term debt. The argument given in Crosbie and Bohn (2003) is that the KMV experience suggests that a typical firm defaults when its asset value falls somewhere between the short-term debt and the total liabilities. We implement the KMV method on the three firms as before. Again, we use daily market capitalizations along with quarterly updated debt levels in 2011 to generate the estimates. The parameter estimates and the DTDs at the year end for the three firms are reported in Panel D of Table 1. The results are very different compared with those by the market value proxy method or the volatility restriction estimation method.

The KMV method represents a clear methodological improvement over the estimation methods discussed thus far. It is self-consistent in the sense that at convergence, the volatility estimate used to produce the implied asset values is also the one implied by the asset values. The KMV method has obvious limitations though. Since its updating mechanism entirely depends on using the implied asset values, one cannot use the method to obtain any unknown parameters present in capital structure. This turns out to be an important limitation for dealing with financial firms, and for which we provide some discussion in the next section and present a practical and better alternative to the KMV method.

In addition to the confusion in the literature mentioned earlier about what the KMV method is, there is also a great deal of misunderstanding about the statistical estimation of the Merton model. For example, Bharath and Shumway (2008) commented in page 1345 on the estimation of the Merton (1974) model with the following statement: "Since the Merton DD model is not a typical econometric model, it is not clear … how its parameters might be estimated with alternative techniques. It is also unclear how standard errors for forecasts can be calculated for the Merton DD model." However, their characterization was inaccurate. In the next section, we will discuss the maximum likelihood technique for this class of models that already appeared in the literature in 1994.

2.4. The Transformed-Data Maximum Likelihood Estimation Method

The transformed-data maximum likelihood estimation (MLE) method for models such as Merton (1974) was proposed in Duan (1994, 2000). When the firm's asset values are not directly observable, one can express the likelihood function of the observed equity values by viewing the equity values as the transformed data where the equity pricing formula in Equation (2) defines the transformation. It should be noted that the transformation involves the unknown asset volatility. By standard transformation theory, the likelihood of observed equity values must equal the product of the likelihood of the asset values (implied by the equity values) and the Jacobian of the inverse transformation (from the equity value back to the asset value). Once the likelihood function is in place, one can apply MLE and its associated statistical inference.

The transformed-data MLE method has been used in the deposit insurance and credit risk literature. For the applications in deposit insurance/banking literature, see for example, Duan and Yu (1994), Duan and Simonato (2002), Laeven (2002) and Lehar (2005). As to its applications in credit analysis, Ericsson and Reneby (2004, 2005) and Wong and Choi (2009) are some examples.

The log-likelihood function based on a sample of n equity prices under the Merton (1974) model can be expressed as:

$$L(\mu,\sigma;S_1,S_2,\ldots,S_n) = -\frac{n-1}{2}\ln(2\pi)-\frac{1}{2}\sum_{t=2}^{n}\ln(\sigma^2 h_t)-\sum_{t=2}^{n}\frac{\hat{W}_t^2}{2\sigma^2 h_t} - \sum_{t=2}^{n}\ln(\hat{V}_t)-\sum_{t=2}^{n}\ln N(d(\hat{V}_t,\sigma,F_t,\tau_t)), \quad (8)$$

where

$$\hat{V}_t = g(S_t;\sigma,F_t,\tau_t),$$
$$\hat{W}_t = \ln\hat{V}_t - \ln\hat{V}_{t-1} - \left(\mu-\frac{1}{2}\sigma^2\right)h_t,$$

$g(.)$ is the inverse of the equity pricing formula in Equation (2), and h_t is the length of time between two consecutive equity values. With h_t, one can easily take care of missing equity values. Maximizing the log-likelihood function in Equation (8) yields maximum likelihood estimators, $\hat{\mu}$ and $\hat{\sigma}$.

Using MLE has clear advantages, because its statistical properties are known and it gives users more than just point estimates. Maximum likelihood estimators are known to be normally distributed in an asymptotic sense. In addition, any differentiable transformation of maximum likelihood estimators is also a maximum likelihood estimator. Thus, the implied asset value can also be characterized by a sampling distribution, and in fact, any quantity of interest can be stated along with a confidence level. Following Duan (1994), the following asymptotic distributions are readily available for the two parameters and the implied asset value at time t:

$$\sqrt{n}\begin{bmatrix}\hat{\mu}-\mu_0\\ \hat{\sigma}-\sigma_0\end{bmatrix}\rightarrow N(0,A_n^{-1})$$
$$\text{where } A_n = -\frac{1}{n}\begin{bmatrix}\frac{\partial L(\hat{\mu},\hat{\sigma})}{\partial\mu^2} & \frac{\partial L(\hat{\mu},\hat{\sigma})}{\partial\mu\partial\sigma}\\ \frac{\partial L(\hat{\mu},\hat{\sigma})}{\partial\sigma\partial\mu} & \frac{\partial L(\hat{\mu},\hat{\sigma})}{\partial\sigma^2}\end{bmatrix} \quad (9)$$

$$\sqrt{n}(\hat{V}_t - V_t)\rightarrow N(0,C_t' A_n^{-1}C_t)$$
$$\text{where } C_t = \begin{bmatrix} 0 \\ \frac{\partial g(St;\hat{\sigma},F_t,\tau_t)}{\partial\sigma}\end{bmatrix} \quad (10)$$

A similar result as in Equation (10) is available for other variables of interest, such as DTD and default probability. In those cases, one should recognize that the parameters affect the sampling error directly as well as indirectly through the implied asset value. Specifically, DTD at time t in accordance with Equation (4) should be viewed as a function of, $\hat{V}_t(\hat{\sigma})$, $\hat{\mu}$ and $\hat{\sigma}$ because these three variables are all subject to sampling variations.

The KMV method was previously compared with the transformed-data MLE method in Duan, Gauthier and Simonato (2004). They argued in the case of the Merton (1974) model that two methods are equivalent both in principle and in implementation, but distinctively different for models involving unknown capital structure parameters. Specifically, they showed by an example of estimation using 250 daily equity values that the parameter estimates and the implied asset value time series produced by the two methods are virtually the same. For example, the KMV method converges to, $\hat{\mu} = -0.025, \hat{\sigma} = 0.177$, and the last implied asset value, V_{250}, equals 0.9708. By the transformed-data MLE method, $\hat{\mu} = -0.025, \hat{\sigma} = 0.175$, and V_{250} is 0.9713. They attributed the minor differences to numerical errors arising from two different numerical algorithms. It turns out that the theoretical argument of Duan *et al*. (2004) is flawed because a complication associated with singularity was overlooked.[5] The numerical difference of two methods become more pronounced when the equity, viewed as a call option, is more out-of-the-money. The difference is evident from our analysis of three firms using the transformed-data MLE method under the same KMV assumption of maturity and default point. The results for IBM and Tokio Marine reported in Panel E of Table 1 by and large confirm the similarity conclusion, because their KMV default points are low relative to their market capitalizations.[6] In the case of Barclays, the KMV default point is quite high and the divergence of two estimation methods is evident.

Putting the similarity/dissimilarity issue aside, the general non-equivalence argument of Duan *et al*. (2004) has important implications on DTD, particularly for financial firms. Taking Barclays as an example, its default point at the end of December 2011, according to the KMV assumption, was 341 billion

GBP (short-term debt plus 50% long-term debt), and its market capitalization was 21 billion GBP. Barclays also had other liabilities in the amount of 1,004 billion GBP, which is huge relative to its market capitalization but is not part of the KMV default point calculation. In contrast, IBM had KMV debt (default point) of 51 billion USD, market capitalization of 217 billion USD, and other liabilities of 29 billion USD. Ignoring other liabilities of IBM may have a minimal effect, but excluding other liabilities of Barclays obviously runs a huge risk of understating its default point, underestimating its implied asset value and overestimating its asset volatility.

It is worth noting one implementation issue generic to all estimation methods. Firms are dynamic entities, responding to environmental changes and also taking initiatives to grow or to consolidate. Over a sample period, large changes in a firm's market capitalization may simply reflect scale changes but not the fundamental nature of its return per unit of assets. Failing to neutralize the effect caused by a scale change on market capitalization is likely to produce an over- or under-statement of the asset return. A scale change, however, inflates asset volatility simply because of those large but artificial moves in the asset values. Duan (2010) and Duan *et al*. (2012) proposed to scale the implied asset value by its corresponding book value. Consider, for example, a firm that has just doubled its asset base without making changes to any other aspect of its operations. This doubling of the asset base should in principle also double its implied asset value, market capitalization, and so on. After being scaled by its book value, however, the asset return will not exhibit an abnormal 100% jump as would happen otherwise. Obviously, if the book asset value stays unchanged throughout the sample period, such scaling has no effect as it should.

2.5. Other Liabilities and the Transformed-Data MLE Estimation Method

To our knowledge, the KMV default point formula is not meant for financial firms. In fact, the research papers in the corporate default/bankruptcy prediction literature tend to exclude financial firms from analysis; for example, Duffie *et al*. (2007). Although the reasons for exclusion are not typically given, it is quite clear from the discussion above that putting financial and non-financial firms into a common data sample requires properly treating financial firms to avoid serious distortions to empirical results. It is also clear that financial firms need a special treatment of their default points. The challenge is how to factor in other liabilities in an operationally feasible way.

To account for other liabilities, a method for their inclusion into the default point was proposed and implemented in Duan (2010) and Duan *et al*. (2012). A haircut is applied to other liabilities much like what the KMV assumption does to the long-term debt. The difference is that the specific haircut is not a predetermined number and will be estimated using the transformed-data MLE method. This estimated haircut method has already been incorporated into the corporate default prediction system under the Credit Research Initiative at the Risk Management Institute, National University of Singapore. The objective of that initiative is to provide objective third-party credit information as a public good through offering freely accessible daily updated default probabilities on the exchange-listed firms around the globe.

The specific treatment of other liabilities involves the following definition of default point (i.e., relevant debt level for triggering default):

$$F = \text{short-term debt} + 0.5 \times \text{long-term debt} + \delta \times \text{other liabilities}$$

where parameter δ defines the haircut. Setting $\delta = 0$, one is back to the KMV default point assumption. Under the above generalized default point assumption, the unknown parameters of the model increase from two (μ and σ) to three (μ, σ and δ) with the first two parameters governing the asset value dynamic and the additional one appearing solely in the capital structure. If one attempts to apply the KMV iterative procedure to estimate δ, it will become clear that δ cannot be updated by the implied asset values, because this parameter only appears in the capital structure. However, it is possible to estimate δ with the transformed-data MLE method because this parameter is part of the Jacobian of the transformation from equity value back to the asset value.

The following log-likelihood function used in Duan (2010) and Duan *et al*. (2012) has scaled the asset value by its corresponding book value:

$$L(\mu,\sigma;S_1,\ldots,S_n) = -\frac{n-1}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=2}^{n}\ln(\sigma^2 h_t) - \sum_{t=2}^{n}\frac{\hat{W}_t^2}{2\sigma^2 h_t} - \sum_{t=2}^{n}\ln\left(\frac{\hat{V}_t}{A_t}\right) - \sum_{t=2}^{n}\ln N\left(d(\hat{V}_t,\sigma,F_t,\tau_t)\right) \quad (11)$$

where

$$\hat{W}_t = \ln\left(\frac{\hat{V}_t}{\hat{V}_{t-1}}\frac{A_{t-1}}{A_t}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)h_t,$$

and $\hat{V}_t = g(S_t;\sigma, F_t, \tau_t)$, and $g(.)$ is again the inverse of the equity pricing formula in Equation (2), A_t is the book asset value, and h_t is still the length of time between two consecutive equity values.

Panel F of Table 1 presents the estimation results on the same three firms that we have been analyzing throughout this paper. The estimates for δ range from 0.46 for IBM to 0.62 for Barclays. The two financial firms (Barclays and Tokio Marine) have remarkably close δ. For an industrial firm like IBM, δ is small and the other liabilities are even smaller relative to the market capitalization. Therefore, using the KMV default point formula does not cause too much distortion. In fact, comparing the volatility estimates for IBM under the KMV method with this method shows a minor difference. In the case of financial firms, the impact is quite big. It is also quite clear that the KMV default point assumption overestimates the asset volatilities of financial firms. The implied asset values in the table suggest that the KMV default point assumption has the effect of lowering their values. The net effect on DTD or DTD* is naturally a mixed one, sometimes higher and other times lower, and the results in the table suggests that the default point assumption indeed affects DTDs.

Since including other liabilities is meant to deal with financial firms, we further analyze its impact by considering more financial firms with two additional banks and two additional insurance companies from different regions of the world. Table 2 reports the estimation results on three banks: Bank of America, Barclays and DBS. Barclays has been presented earlier in Table 1 but is included here for easy comparisons with other banks. Similarly, we present the results of three insurance companies in Table 3 where

Table 2. Banks using two estimation methods.

	Bank of America (Million USD)	**Barclays (Million GBP)**	**DBS (Million SGD)**
Panel A: Input Variables			
Market cap	56,355	21,477	26,976
Short-term debt	617,218	255,193	47,696
Long-term debt	383,517	171,657	18,940
Other liabilities	1,038,408	1,004,083	210,572
Panel B: The KMV Method			
μ	−20.41%	−7.90%	−0.94%
σ	9.41%	6.09%	25.85%
Asset value (12/2011)	849,796	359,291	83,381
DTD (12/2011)	−1.6927	−0.4710	1.2948
DTD* (12/2011)	0.5232	0.8563	1.4604
Panel C: The Transformed-Data MLE Method (Including Other Liabilities)			
μ	−6.45%	−1.02%	−3.88%
σ	3.39%	1.54%	4.51%
δ	57.40%	61.83%	67.24%
Asset value (12/2011)	1,456,323	979,658	224,990
DTD (12/2011)	−0.8484	0.5292	1.8795
DTD* (12/2011)	1.0579	1.1915	2.7491

Table 3. Insurance companies using two estimation methods.

	Sun Life (Million CAD)	AXA SA (Million EUR)	Tokio Marine (Million JPY)
Panel A: Input Variables			
Market cap	11,046	23,678	1,371,714
Short-term debt	694	103,590	1,922,395
Long-term debt	4,889	9,601	121,673
Other liabilities	188,766	543,661	12,095,019
Panel B: The KMV Method			
μ	−0.4061	−0.2891	−0.2690
σ	0.2389	0.4078	0.1684
Asset value (12/2011)	14,154	118,351	3,352,857
DTD (12/2011)	4.4863	−0.6972	1.4360
DTD* (12/2011)	6.3062	0.2156	3.1174
Panel C: The Transformed-Data MLE Method (Including Other Liabilities)			
μ	−0.0365	−0.0365	−0.0508
σ	0.0350	0.0784	0.0517
δ	0.5828	0.6172	0.6078
Asset value (12/2011)	122,856	460,403	10,696,331
DTD (12/2011)	1.3611	−0.0341	1.6400
DTD* (12/2011)	2.4077	0.4645	2.6339

Sun Life and AXA SA are new but Tokio Marine was previously reported. The input variables in Panel A of these two tables are given in million units of local currencies at the end of December 2011. Panels B and C reports the estimation results using the daily market capitalizations and quarterly updated financial statements in 2011. Panel B is for the KMV method whereas Panel C is for the transformed-data MLE method described in this section. The results in these tables confirm our earlier conclusion on financial firms; that is, other liabilities play an important role in their credit risk analysis. Moreover, the haircut to other liabilities in setting an appropriate default point at the end of 2011 seems to vary little in this sample of banks and insurance companies, even though they are based in various domiciles.

In order to see the effect of estimation method over different phases of the macro environment, we repeat two estimation methods on Tokio Marine over the period from January 2004 to December 2011. We perform estimation once per month and use a one-year moving window of daily data. The plots in Figures 1–4 are the monthly time series of estimates (implied asset values, asset volatilities, DTDs and DTD*s) for the KMV method and the transformed-data MLE method that factors in other liabilities. The KMV method consistently yields lower implied asset values as shown in Figure 1 and higher asset volatilities as in Figure 2. For DTD or DTD* reported in Figures 3 and 4, the results are not as clear cut with the KMV method generating higher DTDs in the earlier period but lower DTDs in some months in the later period. Comparing Figures 3 and 4 clearly shows that DTD* is more stable over time than is DTD, confirming an earlier assertion about large sampling errors associated with the estimate for parameter μ.

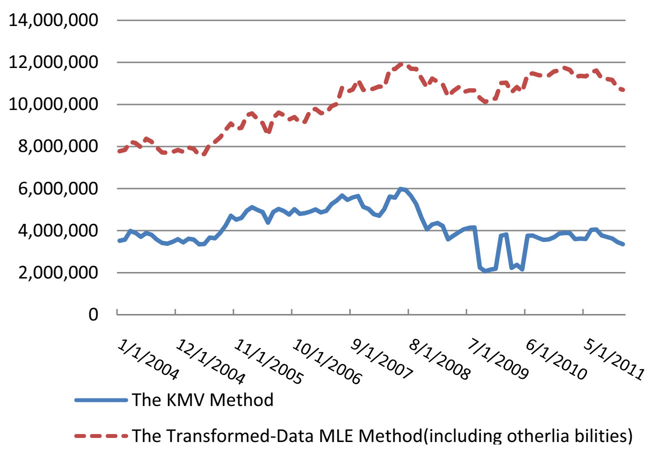

Figure 1. Monthly time series of implied asset values for Tokio Marine under two estimation methods.

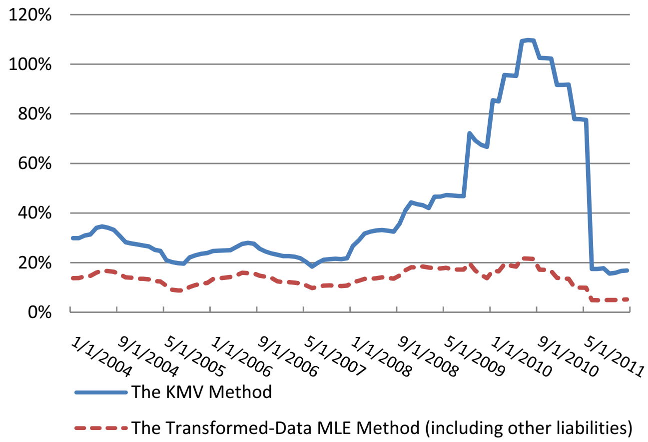

Figure 2. Monthly time series of asset volatilities for Tokio Marine under two estimation methods.

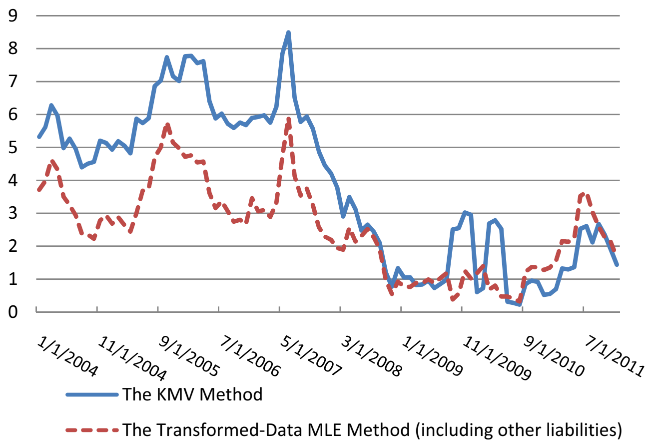

Figure 3. Monthly time series of DTDs for Tokio Marine under two estimation methods.

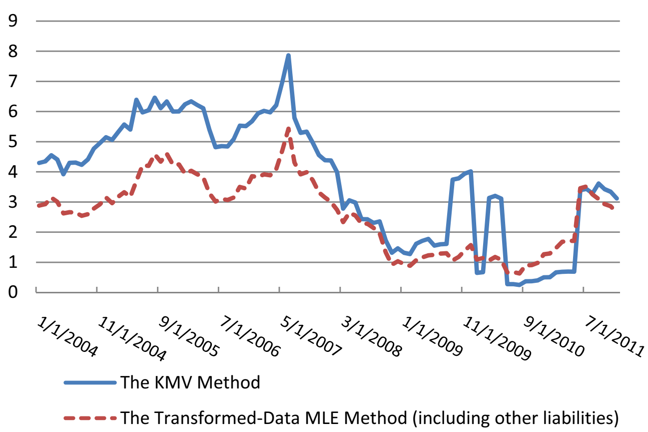

Figure 4. Monthly time series of DTD*s for Tokio Marine under two estimation methods.

The transformed-data MLE method (including other liabilities) has been previously implemented to obtain DTDs in a default study of US firms by Duan *et al*. (2012). In that study, all exchange-listed firms are put into the data sample, which of course includes financial firms. They proposed a forward-intensity default prediction model to link defaults to some common risk factors shared by all firms, such as a benchmark interest, and individual firm attributes, such as liquidity and DTD. With the properly computed DTDs, they are able to show that their default prediction model, after being fitted to the whole data sample, can perform equally well in the financial and non-financial subsectors, suggesting that incorporating other liabilities with a haircut into the default point is a productive way of dealing with financial firms that are typically of high leverage.

CONCLUSION

In this article, we introduce a popular credit risk measure known as DTD, describe its theoretical foundation and usage, and review the ways of implementation available in the literature. We cover the market value proxy method, the volatility restriction method, the KMV method and the transformed-data MLE method. Three types of firms are used to illustrate the implementation of these methods, and through which we gain better understanding of the methods' strengths and weaknesses. We pay special attention to financial firms (banks and insurance companies) because of their economic importance and uniqueness in capital structure. Financial firms typically have higher leverage than non-financial firms, and the popular KMV estimation seems ill-suited for this category of firms. Blindly applying the KMV method is shown to cause serious distortion to credit analysis. We then introduce a recent advancement in the estimation of the Merton (1974) credit risk model, a method that specifically accounts for the high-leverage feature of financial firms in the transformed-data MLE framework of Duan (1994). The new approach is shown, through an analysis of three banks and three insurance companies from different regions of the globe, to differ from the KMV method in a material way. We contend that the new estimation method is superior compared with existing methods. This new method is also a practical technology because it has been incorporated into the default prediction system under the non-profit Credit Research Initiative at the Risk Management Institute of National University of Singapore to generate daily updated default probabilities on exchange-listed firms around the globe.[7]

NOTES

1 For example, Bharath and Shumway (2008) concluded that DTD is not a sufficient statistic for default prediction. Duffie *et al*. (2007) and Duan *et al*. (2012) showed that in addition to DTD, there are other variables significantly contributing to default prediction.

2 DTD^* is similar to the DTD in the KMV method, which uses a formula, i.e., Equation (5) of Crosbie and Bohn (2003), slightly different from Equation (5) of this paper. The KMV DTD replaces $\ln(V_t/F)$ with its approximation $(V_t - F)/F$.

3 The method by Jones, Mason and Rosenfeld (1984) only uses Equation (7). The asset value is obtained by a different means.

4 Because of the exposition in Crosbie and Bohn (2003), the KMV method has been misunderstood by some as a two-equation volatility restriction method; for example, Chapter 11 of Caouette *et al*. (2008) and Footnote # 8 of Eisdorfer and Hsu (2011). In fact, they, along with others such as Bharath and Shumway (2008) also failed to recognize that that the two-equation volatility restriction method was first proposed by Ronn and Verma (1986) and has been widely applied in the deposit insurance literature.

5 Singularity arises from casting the transformed-data problem in the EM algorithm framework. One can create an incomplete-data estimation problem by viewing the observed equity price as the sum of the value produced by Equation (2) and some measurement error, and conduct a maximum likelihood estimation using the EM algorithm. Then by shrinking the measurement error, one is in effect back to the original estimation problem. There is a problem with this reasoning, however. Since the measurement error's density function, a part of the complete-data likelihood, approaches the Dirac delta function as one shrinks the measurement error to zero (i.e., singularity), it dominates all other terms in the complete-data likelihood, and thus cannot be ignored. Consequently, one cannot be certain that the KMV method produces the maximum likelihood estimate.

6 Note that under the KMV assumption, Tokio Marine's other liabilities are not included in the default point calculation. In other words, its equity is much more in-the-money than what it would otherwise be.

7 Details on the non-profit Credit Rating Initiative of Risk Management Institute can be found at http://rmicri.org.

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<u>A Post-publication Clarification Note</u>
"Measuring Distance-to-Default for Financial and Non-Financial Firms"
by J.C. Duan and T. Wang, *Global Credit Review* 2, 2012, p.95-108

We introduced in Section 2.5 the transformed-data MLE estimation method when a firm's other liabilities are factored in the estimation of its distance-to-default (DTD). The results for the examples presented in the paper under this case were actually taken directly from the Credit Research Initiative database at the Risk Management Institute (RMI) of National University of Singapore. The specific numbers for parameter estimates and DTD were computed with the estimation method implemented per its Technical Report (Version 2011, Update 1) whereas DTD* were computed according to Addendum 4 to the Technical Report.

The way that RMI calculates the DTD involves two passes. In the first pass, it estimates a firm's three free parameters – μ, σ and δ. Optimization is performed on μ and σ without constraints, but δ is constrained to the unit interval [0, 1] in the first instance. Estimations are conducted monthly over the time series, and a moving sample of daily prices for two years is used. All subsequent estimations are subjected to a different interval constraint of [max(0, $\hat{\delta}_{n-1} - 0.05$), min(1, $\hat{\delta}_{n-1} + 0.05$)], where $\hat{\delta}_{n-1}$ is the estimate of δ obtained in the previous month. In other words, a ten-percent band is placed around the previous estimate and the band is floored by 0 and capped at 1.

The estimation is repeated for all firms in an economy. In the second pass, an economy is divided into financial and non-financial sectors, and for each sector, an average δ is obtained and used for all firms in that sector for that particular monthly estimation. For the second pass, the moving sample is shortened to one year. With the average δ in place, one then uses the same estimation method for each firm to obtain the remaining two parameters: μ and σ.

The motivation for the RMI implementation is to obtain more stable parameter estimates. Because financial statements are available at best quarterly, there will never be more than three changes in the balance sheet over one year. Changes in the elements of the default point formula are critical to the identification of δ, because the asset value under the model specification is latent. Since any scale shift in the default point can be absorbed by a compensating change in the latent asset value, there needs to be at least one change in the components of the default point over the sample to pin down δ, and more changes are of course better. A popular choice of estimation window for the DTD calculation is one year. Using a two-year moving window in the first pass strikes a balance between more and timely information. Switching back to the one-year moving window in the second pass is to conform to the standard practice. For further details on RMI's DTD and DTD* implementation, readers are referred to the aforementioned RMI Technical Report (Section 3.2, page 79-81) and Addendum 4 to the Technical Report.